SOBOLEV MAPPINGS: LIPSCHITZ DENSITY IS NOT AN ISOMETRIC INVARIANT OF THE TARGET

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ABSTRACT. If M is a compact smooth manifold and X is a compact metric space, the Sobolev space $W^{1,p}(M,X)$ is defined through an isometric embedding of X into a Banach space. We prove that the answer to the question whether Lipschitz mappings $\mathrm{Lip}\,(M,X)$ are dense in $W^{1,p}(M,X)$ may depend on the isometric embedding of the target.

1. INTRODUCTION

Sobolev mappings between manifolds $W^{1,p}(M,N)$ play a fundamental role in geometric variational problems like the theory of harmonic mappings. Eells and Lemaire [7], asked whether smooth mappings are dense in $W^{1,p}(M,N)$ and it turns out that the answer to that question depends on the topology of the manifolds, see e.g. [2], [3], [9], [19], [20], for early results. Finally a necessary and sufficient condition was discovered in Hang and Lin, [14]. The theory of Sobolev mappings between manifolds has been extended to the case of mappings into metric spaces. Research in that direction was initiated in the work of Ambrosio, [1], Gromov and Schoen, [8], Korevaar and Schoen, [17], Capogna and Lin [4] and Reshetnyak [18] just to name a few. Finally the theory was even extended to the case of Sobolev mappings between metric spaces, see Heinonen and Koskela [15] and Heinonen, Hoskela, Shanmugalingam and Tyson [16]. It is natural to inquire what would be suitable generalizations of density results known in the case of mappings between smooth manifolds to the case of a metric target. This problem was explicitly formulated in the work of Heinonen, Koskela, Shanumgalingam and Tyson [16, Remark 6.9] and some partial results have been obtained in [5], [10] and [11]. For a more detailed introduction to the subject, see the survey paper [12].

In this paper we consider the Sobolev space $W^{1,p}(M,X)$ of mappings from a smooth compact Riemannian manifold (with or without boundary) into a compact metric space X. Every metric space admits an isometric embedding into a Banach space; if X is separable (in particular if X is compact) it can

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be isometrically embedded into ℓ^∞ (the Kuratowski embedding). The space $\ell^\infty = (\ell^1)^*$ is dual to a separable Banach space. Thus we may assume that a compact space X is isometrically embedded into a Banach space V, $X \subset V$, where $V = Y^*$ is dual to a separable Banach space Y. In what follows we assume that every Banach space into which we embed X is a dual to separable Banach space. The vector valued Sobolev space $W^{1,p}(M,V)$ is a Banach space and we may define

$$W^{1,p}(M,X) = \left\{ f \in W^{1,p}(M,V) :\, f(x) \in X \text{ a.e.} \right\} .$$

If X is compact, then all mappings into X are bounded as mappings into V and therefore integrable. The compactness assumption is to avoid problems with integrability of the mapping.

The space $W^{1,p}(M,X)$ is equipped with a metric inherited from the norm of $W^{1,p}(M,V)$ and we may inquire whether for a given metric space X, Lipschitz mappings $\mathrm{Lip}\,(M,X)$ are dense in $W^{1,p}(M,X)$. This is exactly the problem that was formulated in [16, Remark 6.9]. It turns out that Sobolev mappings $W^{1,p}(M,X)$ can be defined in an intrinsic way independent of the isometric embedding, see Proposition 2.5. However, the metric in $W^{1,p}(M,X)$ does depend on the embedding. A simple example is provided in [10, p. 438]. Thus if $\lambda : X \to V$ is an isometric embedding of X into V, used to define the metric in $W^{1,p}(M,X)$, we should rather denote the space by $W^{1,p}_\lambda(M,X)$, but in practice, the subscript λ is often omitted. Regarding the space of Lipschitz mappings $\mathrm{Lip}\,(M,X)$ there is no need to use subscript λ. The following question is very natural:

Question. *Does the answer to the question about the density of Lipschitz mappings in $W^{1,p}(M,X)$ depend on the isometric embedding of X into a Banach space?*

This problem arose soon after the publication of [16] and it was explicitly formulated in [10, Question 1]. The following result gives a partial answer to that problem.

Theorem 1.1. *Let M be a compact Riemannian manifold, X a compact metric space and $1 \leq p < \infty$. If for some isometric embedding $\lambda : X \to V$ of X into a Banach space V Lipschitz mappings $\mathrm{Lip}\,(M,X)$ are dense in $W^{1,p}_\lambda(M,X)$ in the following strong sense: for every $f \in W^{1,p}_\lambda(M,X)$ and every $\varepsilon > 0$ there is $g \in \mathrm{Lip}\,(M,X)$ such that*

$$|\{x :\, f(x) \neq g(x)\}| < \varepsilon \quad \textit{and} \quad \|f-g\|_{1,p} < \varepsilon\,,$$

then Lipschitz mappings are dense in $W^{1,p}_\nu(M,X)$ for any other isometric embedding $\nu : X \to W$ of X into a Banach space W.

As explained above, we assume here that each of the Banach spaces V and W is dual to a separable Banach space. This result is a version of [11, Theorem 4]. The density in the strong sense seems a typical property. The following result was proved in [11, Lemma 13].

Proposition 1.2. *If a Banach space V is dual to a separable Banach space and $f \in W^{1,p}(M, V)$, then for every $\varepsilon > 0$ there is $g \in \mathrm{Lip}\,(M, V)$ such that $|\{x : f(x) \neq g(x)\}| < \varepsilon$ and $\|f - g\|_{1,p} < \varepsilon$.*

In view of the two results one could naturally expect that the answer to the above question should be that the density is not depend on the choice of the isometric embedding and that is a completely incorrect intuition. Indeed, the following theorem, the main result of the paper, provides an example that gives a different answer.

Theorem 1.3. *There is a compact and connected set $X \subset \mathbb{R}^{n+2}$ such that Lipschitz mappings $\mathrm{Lip}\,(S^n, X)$ are dense in $W^{1,n}(S^n, X)$, while if $\kappa : X \to \ell^\infty$ is the Kuratowski embedding, then Lipschitz mappings $\mathrm{Lip}\,(S^n, X)$ are not dense in $W^{1,n}_\kappa(S^n, X)$.*

Since X is a subset of $\mathbb{R}^{n+2}$ in the first definition of $W^{1,n}(S^n, X)$ we just consider the identity embedding of X into $\mathbb{R}^{n+2}$ (which is a Banach space) and we avoid the subscript id.

The paper is organized as follows. In Section 2 we review the theory of Sobolev mappings into metric spaces. The main result of the section is Corollary 2.7 which is one of the main tools in the proof of Theorem 1.3. In Section 3 we prove Theorem 1.1 and Section 4 is devoted to the proof of Theorem 1.3.

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2. Sobolev mappings into metric spaces

In this section we briefly discuss the construction of the Sobolev space of mappings from a manifold into a metric space. We follow the presentation given in [13]. We refer the reader to that paper for detailed proofs and references. For simplicity we formulate most of the definitions and results for open subsets $\Omega \subset \mathbb{R}^n$ instead of compact manifolds, but the generalization to the manifold case is straightforward via the use of local coordinate systems. Most of the results in this section are known. The new results are Lemma 2.6 and Corollary 2.7.

If V is any Banach space (not necessarily dual to a separable Banach space) and $A \subset \mathbb{R}^n$ is (Lebesgue) measurable, we say that $f \in L^p(A, V)$ if

(1) f is *essentially separably valued*: $f(A\setminus Z)$ is a separable subset of V for some set Z of Lebesgue measure zero,
(2) f is *weakly measurable*: for every $v^*\in V^*$ with $\|v^*\|\leq 1$, $\langle v^*,f\rangle$ is measurable,
(3) $\|f\|\in L^p(A)$.

If $f\in L^1(A,V)$ we define the integral

$$\int_A f(x)\,dx\in V$$

in the Bochner sense, see [21, Chapter 5, Sections 4-5], [6]. The Bochner integral has two important properties: For every $v\in V^*$

$$\left\langle v^*,\int_A f(x)\,dx\right\rangle=\int_A\langle v^*,f(x)\rangle\,dx$$

and

$$\left\|\int_A f(x)\,dx\right\|\leq\int_A||f(x)||\,dx.$$

Definition 2.1. Let $\Omega\subset\mathbb{R}^n$ be an open set and V any Banach space (not necessarily dual). The Sobolev space $W^{1,p}(\Omega,V)$, $1\leq p<\infty$, is defined as the class of all functions $f\in L^p(\Omega,V)$ such that for $i=1,2,\ldots,n$ there is $f_i\in L^p(\Omega,V)$ such that for every $\varphi\in C_0^\infty(\Omega)$

$$\int_\Omega\frac{\partial\varphi}{\partial x_i}\,f=-\int_\Omega\varphi f_i\,,$$

where the integrals are taken in the sense of Bochner (note that the integrands are supported on compact subsets of Ω). We denote $f_i=\partial f/\partial x_i$ and call these functions *weak partial derivatives* of f. We also write $\nabla f=(\partial f/\partial x_1,\ldots,\partial f/\partial x_n)$ and

$$|\nabla f|=\left(\sum_{i=1}^n\left\|\frac{\partial f}{\partial x_i}\right\|^2\right)^{1/2}. \tag{2.1}$$

Sometimes we will write $|\nabla f|_V$ to emphasize the Banach space with respect to which we compute the length of the gradient. The space $W^{1,p}(\Omega,V)$ is equipped with the norm

$$\|f\|_{1,p}=\left(\int_\Omega\|f\|^p\right)^{1/p}+\left(\int_\Omega|\nabla f|^p\right)^{1/p}.$$

It is easy to prove that $W^{1,p}(\Omega,V)$ is a Banach space.

It easily follows from the definition (see [13, Proposition 2.3]) that for every $v^*\in V^*$ with $\|v^*\|\leq 1$, we have $\langle v^*,f\rangle\in W^{1,p}(\Omega)$ and

$$|\nabla\langle v^*,f\rangle|\leq|\nabla f|\quad\text{a.e.} \tag{2.2}$$

Observe that $v^*:V\to\mathbb{R}$ is a 1-Lipschitz function and it turns out that under the additional assumption that V is dual to a separable Banach space

(2.2) holds with v^* replaced by any 1-Lipschitz function. Moreover $|\nabla f|$ is, in a certain sense, the best lower bound for $|\nabla\langle v^*, f\rangle|$. Namely we have.

Proposition 2.2. *Let $\Omega \subset \mathbb{R}^n$ be open, let V be dual to a separable Banach space and let $1 \le p < \infty$. Then for $f \in W^{1,p}(\Omega, V)$ we have*

(1) *If $0 \le g \in L^p(\Omega)$ is such that for every $v^* \in V^*$ with $\|v^*\| \le 1$ we have*

$$|\nabla\langle v^*, f\rangle| \le g \quad a.e.$$

then

$$|\nabla f| \le Cg \quad a.e.$$

(2) *For every 1-Lipschitz function $\varphi : V \to \mathbb{R}$, we have $\varphi \circ f \in W^{1,p}(\Omega)$ and*

$$|\nabla(\varphi \circ f)| \le |\nabla f|.$$

The first part of this result follows from Theorem 2.14 and Lemma 2.13 in [13]. The second part is a consequence of the estimate (2.2) and the proof of Proposition 2.16 in [13].

The proof of the above result is based on the characterization of Sobolev mappings by absolute continuity on lines. For the following result, see [13, Lemma 2.8 and Lemma 2.13].

Lemma 2.3. *Let $V = Y^*$ be dual to a separable Banach space Y. If $f : [a,b] \to V$ is absolutely continuous, then the limit*

$$g(x) := \lim_{h\to 0} \left\| \frac{f(x+h) - f(x)}{h} \right\|$$

exists a.e. and $g \in L^1([a,b])$. Moreover for a.e. $x \in (a,b)$ there is a vector $f'(x) \in V$ such that $\|f'(x)\| \le g(x)$ and

$$\left\langle v^*, \frac{f(x+h) - f(x)}{h} \right\rangle \to \langle v^*, f'(x)\rangle \qquad as\ h \to 0$$

for all $v^* \in Y$. We call $f'(x)$ the w^*-derivative of f at x.

The lemma leads to the following characterization of the Sobolev space, see [13, Lemma 2.12, Lemma 2.13 and Theorem 2.14].

Proposition 2.4. *Let $\Omega \subset \mathbb{R}^n$ be an open set, let $V = Y^*$ be dual to a separable Banach space Y and let $1 \le p < \infty$. Let $f \in L^p(\Omega, V)$. Then $f \in W^{1,p}(\Omega, V)$ if and only if f is absolutely continuous on compact intervals in $\ell \cap \Omega$ for almost all lines ℓ parallel to coordinate axes (possibly after being redefined on a set of measure zero) and if w^*-partial derivatives of f belong to $L^p(\Omega, V)$. Moreover the w^*-partial derivatives are equal to weak derivatives of f.*

Now we are ready to define the Sobolev space of mappings with values into a compact metric space X. If $\lambda : X \to V$ is an isometric embedding of X into a Banach space V which is dual to a separable Banach space, then we define

$$W^{1,p}(\Omega, X) = W^{1,p}_\lambda(\Omega, X) = \{f \in W^{1,p}(\Omega, V) : \, f(x) \in \lambda(X) \text{ a.e.}\}.$$

Every compact (or even separable) metric space (X, d) admits an isometric embedding into ℓ^∞. Indeed, if $\{x_i\}_{i=1}^\infty$ is a dense subset of X and $x_0 \in X$ any point, then one can easily show that the mapping

$$\kappa : X \to \ell^\infty, \qquad \kappa(x) = \big(d(x, x_i) - d(x_0, x_i)\big)_{i=1}^\infty$$

is the isometric embedding. It is the well known *Kuratowski embedding.* Therefore the Sobolev space $W^{1,p}(\Omega, X)$, can be defined for any compact metric space X, because we can always use the Kuratowski embedding. Moreover observe that $\ell^\infty = (\ell^1)^*$, so the space ℓ^∞ is dual to a separable Banach space.

It turns out that the Sobolev space of mappings into X can be defined in an intrinsic way independent of the choice of the embedding λ. For the following result see Definition 1.2 and argument on p. 698 in [13].

Proposition 2.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, X a compact metric space and $1 \le p < \infty$. Then $f \in W^{1,p}(\Omega, X)$ if and only if there is a nonnegative function $g \in L^p(\Omega)$ such that for every Lipschitz function $\varphi : X \to \mathbb{R}$, $\varphi \circ f \in W^{1,p}(\Omega)$ and $|\nabla(\varphi \circ f)| \le \mathrm{Lip}\,(\varphi) g$ a.e. Here $\mathrm{Lip}\,(\varphi)$ stands for the Lipschitz constant of φ.*

Let $f = (f^i)_{i=1}^\infty \in W^{1,p}(\Omega, \ell^\infty)$. Then f is absolutely continuous on almost all lines parallel to coordinate axes. Since the weak partial derivatives are equal to w^*-partial derivatives we can compute them with help of Lemma 2.3.

Let $\delta_i \in (\ell^\infty)^*$ be defined as ith coordinate, i.e.

$$\delta_i(z_1, z_2, \ldots) = z_i,$$

Hence $f^i = \delta_i \circ f$ is absolutely continuous on almost all lines as a composition of f with the Lipschitz function δ_i. Thus the coordinate functions f^i belong to $W^{1,p}(\Omega)$.

For $k = 1, 2, \ldots, n$, Lemma 2.3 gives a formula for weak partial derivatives

$$\frac{\partial f}{\partial x_k} = \left(\left(\frac{\partial f}{\partial x_k} \right)^i \right)_{i=1}^\infty \in L^p(\Omega, \ell^\infty)\,.$$

Indeed,

$$\frac{f^i(x + he_k) - f^i(x)}{h} = \left\langle \delta_i, \frac{f(x + he_k) - f(x)}{h} \right\rangle \to \left\langle \delta_i, \frac{\partial f}{\partial x_k}(x) \right\rangle = \left(\frac{\partial f}{\partial x_k} \right)^i$$

a.e., that is

$$\left(\frac{\partial f}{\partial x_k}(x)\right)^i = \frac{\partial f^i}{\partial x_k}(x) \quad \text{a.e.}$$

The next two results are new.

Lemma 2.6. *Let $f, g : [a,b] \to \mathbb{R}^N$ be absolutely continuous and let $\kappa : \mathbb{R}^N \to \ell^\infty$ be the Kuratowski embedding. Then $\bar{f} = \kappa \circ f$ and $\bar{g} = \kappa \circ g$ are absolutely continuous functions with values into ℓ^∞ and the w^*-derivative $(\bar{f} - \bar{g})' : [a,b] \to \ell^\infty$ satisfies*

$$\|(\bar{f} - \bar{g})'(t)\|_\infty \geq \max\{|f'(t)|, |g'(t)|\} \geq \frac{1}{2}\left(|f'(t)| + |g'(t)|\right)$$

for almost every $t \in [a,b]$ such that $f(t) \neq g(t)$.

Proof. Let $\{x_i\}_{i=1}^\infty \subset \mathbb{R}^N$ be a dense subset, $x_0 \in \mathbb{R}^N$ and let $\kappa : \mathbb{R}^N \to \ell^\infty$ be the Kuratowski embedding. It is easy to see that

$$(\bar{f} - \bar{g})(s) = (|f(s) - x_i| - |g(s) - x_i|)_{i=1}^\infty$$

and

$$(\bar{f} - \bar{g})'(s) = \left(|f(s) - x_i|' - |g(s) - x_i|'\right)_{i=1}^\infty$$

for a.e. $s \in [a,b]$. Fix $t \in [a,b]$ such that $f(t) \neq g(t)$ and both f and g are differentiable at t. If $f'(t) = g'(t) = 0$, the inequality is obvious. Assume, then that one of the derivatives is non zero, say $f'(t) \neq 0$. Let $\delta > 0$. If we choose x_i to be very close to $f(t) + f'(t)\delta$, then the function $s \mapsto |f(s) - x_i|$ is decreasing near $s = t$ at the rate very close to $|f'(t)|$, because the point x_i is nearly exactly in the direction in which $f(s)$ is going. More precisely

$$|f(t) - x_i|' = \frac{f'(t) \cdot (f(t) - x_i)}{|f(t) - x_i|} = |f'(t)| \cos\theta,$$

where θ is the angle between the vectors $f'(t)$ and $f(t) - x_i$. If x_i is very close to $f(t) + f'(t)\delta$, θ is very close to π. Hence given $\varepsilon > 0$ and $\delta > 0$ we can find x_i so close to $f(t) + f'(t)\delta$ that

$$|f(t) - x_i|' \leq -|f'(t)| + \varepsilon.$$

Choosing $x_{i'}$ close to $f(t) - f'(t)\delta$ we make the function $s \mapsto |f(s) - x_{i'}|$ increasing at the rate very close to $|f'(t)|$, so given $\varepsilon > 0$ and $\delta > 0$ we can find $x_{i'}$ so close to $f(t) - f'(t)\delta$ that

$$|f(t) - x_{i'}|' \geq |f'(t)| - \varepsilon.$$

Since $|f(t) - g(t)| > 0$ (remember that we assume that $f(t) \neq g(t)$), taking $\delta > 0$ sufficiently small we can make the points x_i and $x_{i'}$ so close to $f(t)$ that

$$\left||g(t) - x_i|' - |g(t) - x_{i'}|'\right| < \varepsilon.$$

Hence either

$$\left||f(t) - x_i|' - |g(t) - x_i|'\right| \geq |f'(t)| - 2\varepsilon$$

or

$$\big||f(t)-x_{i'}|'-|g(t)-x_{i'}|'\big|\geq|f'(t)|-2\varepsilon$$

by the triangle inequality. Thus

$$\|(\bar{f}-\bar{g})'(t)\|_\infty=\sup_j\big||f(t)-x_j|'-|g(t)-x_j|'\big|\geq|f'(t)|-2\varepsilon. \tag{2.3}$$

Since the inequality is true for any $\varepsilon>0$ we have

$$\|(\bar{f}-\bar{g})'(t)\|_\infty\geq|f'(t)|.$$

If $g'(t)=0$ the lemma follows. If $g'(t)\neq 0$ we can repeat the above argument with f replaced by g and obtain the estimate

$$\|(\bar{f}-\bar{g})'(t)\|_\infty\geq|g'(t)|.$$

The two estimates combined together prove the lemma. □

Corollary 2.7. *Let $\Omega\subset\mathbb{R}^n$ be bounded. Let $f,g\in W^{1,p}(\Omega,\mathbb{R}^N)$ and let $\kappa:\mathbb{R}^N\to\ell^\infty$ be the Kuratowski embedding. Then $\bar{f}=\kappa\circ f, \bar{g}=\kappa\circ g\in W^{1,p}(\Omega,\ell^\infty)$ and*

$$|\nabla(\bar{f}-\bar{g})|\geq C(n)\,(|\nabla f|+|\nabla g|)\,\chi_{\{f\neq g\}}\quad a.e.$$

Proof. The result follows immediately from the fact that f and g are absolutely continuous almost all lines parallel to the coordinate directions, from Lemma 2.3 and from the definition

$$|\nabla(\bar{f}-\bar{g})|=\left(\sum_{k=1}^n\left\|\frac{\partial(\bar{f}-\bar{g})}{\partial x_k}\right\|_\infty^2\right)^{1/2}.$$

The proof is complete. □

3. Proof of Theorem 1.1

The following result is well known in the case of real valued Sobolev functions, but since the vector valued case is more delicate, we provide a short proof.

Lemma 3.1. *Let $V=Y^*$ be dual to a separable Banach space, let $\Omega\subset\mathbb{R}^n$ be an open set and let $1\leq p<\infty$. If the functions $f_1,f_2\in W^{1,p}(\Omega,V)$ are equal on a measurable set E, then $\nabla f_1=\nabla f_2$ a.e. on E.*

Proof. Let $f=f_1-f_2$. Then $f=0$ on E and we need to prove that $\nabla f=0$ a.e. on E. Let $v^*\in Y$. Then $\langle v^*,f\rangle\in W^{1,p}(\Omega)$. Since $\langle v^*,f\rangle=0$ on E it is well known that $\nabla\langle v^*,f\rangle(x)=0$ for all $x\in E\setminus Z_{v^*}$ for some set Z_{v^*} of Lebesgue measure zero. Let $D\subset Y$ be a countable dense set. The set $Z=\bigcup_{v^*\in D}Z_{v^*}$ has measure zero and $\nabla\langle v^*,f\rangle(x)=0$ for all $x\in E\setminus Z$ and all $v^*\in D$. The weak partial derivatives of f are equal to w^*-partial derivatives, see Proposition 2.4. Let f be absolutely continuous on $\ell \cap \Omega$, where ℓ is parallel to the kth coordinate axis. Then by Lemma 2.3 for $v^* \in Y$

$$\left\langle v^*, \frac{f(x+he_k)-f(x)}{h} \right\rangle \to \left\langle v^*, \frac{\partial f}{\partial x_k}(x) \right\rangle \quad \text{for a.e. } x \in \ell \cap \Omega.$$

On the other hand the limit equals zero if $x \in \ell \cap (E \setminus Z)$ and $v^* \in D$, so

$$\left\langle v^*, \frac{\partial f}{\partial x_k}(x) \right\rangle = 0 \quad \text{for } v^* \in D \text{ and } x \in \ell \cap (E \setminus Z).$$

Hence $\partial f / \partial x_k(x) = 0$ for a.e. $x \in \ell \cap E$ by density of D in Y. □

Let $\nu : X \to W$ be an isometric embedding and $f \in W^{1,p}_\nu(M, X)$. Then $\bar{f} = \lambda \circ \nu^{-1} \circ f \in W^{1,p}_\lambda(M, X)$, because by Proposition 2.5, the Sobolev space of mappings into X can be defined independently of the isometric embedding. It follows from the assumptions of the theorem that there is a sequence of Lipschitz mappings $\bar{g}_k \in \mathrm{Lip}\,(M, \lambda(X))$ such that

$$|\{x : \bar{f}(x) \neq \bar{g}_k(x)\}| \to 0 \quad \text{and} \quad \|\bar{f} - \bar{g}_k\|_{1,p} \to 0 \quad \text{as } k \to \infty.$$

Then $g_k = \nu \circ \lambda^{-1} \circ \bar{g}_k \in \mathrm{Lip}\,(M, \nu(X))$ and

$$|\{x : f(x) \neq g_k(x)\}| = |\{x : \bar{f}(x) \neq \bar{g}_k(x)\}| \to 0 \quad \text{as } k \to \infty.$$

In particular $g_k \to f$ in $L^p(M, W)$, because X is a bounded subset of W. It remains to estimate the gradients.

It follows from Lemma 3.1 that $\nabla f = \nabla g_k$ a.e. on the set where $f = g_k$, so

$$\begin{aligned} \left(\int_M |\nabla f - \nabla g_k|_W^p \, dx \right)^{1/p} &= \left(\int_{\{f \neq g_k\}} |\nabla f - \nabla g_k|_W^p \, dx \right)^{1/p} \\ &\leq \left(\int_{\{f \neq g_k\}} |\nabla f|_W^p \, dx \right)^{1/p} + \left(\int_{\{f \neq g_k\}} |\nabla g_k|_W^p \, dx \right)^{1/p}. \end{aligned}$$

The first integral on the right hand side converges to zero, because $|\{f \neq g_k\}| \to 0$ and we need to show that the second integral converges to zero as well. For $w^* \in W^*$ with $\|w^*\| \leq 1$ the function

$$v \mapsto \langle w^*, \nu(\lambda^{-1}(v)) \rangle$$

is 1-Lipschitz continuous on $\lambda(X) \subset V$ and hence it extends to a 1-Lipschitz continuous function $\varphi : V \to \mathbb{R}$ (McShane extension). Since

$$\langle w^*, g_k(x) \rangle = (\varphi \circ \bar{g}_k)(x), \quad x \in M$$

Proposition 2.2 gives

$$|\nabla \langle w^*, g_k \rangle| = |\nabla(\varphi \circ \bar{g}_k)| \leq |\nabla \bar{g}_k|_V \quad \text{a.e.}$$

Then another application of Proposition 2.2 yields

$$|\nabla g_k|_W \leq C |\nabla \bar{g}_k|_V \quad \text{a.e.}$$

Hence

$$\left(\int_{\{f\neq g_k\}}|\nabla g_k|^p_W\,dx\right)^{1/p} \leq C\left(\int_{\{f\neq g_k\}}|\nabla \bar{g}_k|^p_V\,dx\right)^{1/p}$$

$$\leq \; C\left(\left(\int_{\{f\neq g_k\}}|\nabla \bar{f}-\nabla\bar{g}_k|^p_V\,dx\right)^{1/p}+\left(\int_{\{f\neq g_k\}}|\nabla \bar{f}|^p_V\,dx\right)^{1/p}\right)\to 0$$

as $k\to\infty$. The proof is complete. $\square$

4. Proof of the main Theorem 1.3

We begin with the construction of the set X. Actually this is exactly the same set as in [10] where it was used to provide a counterexample to a different question. In the description below we will try to emphasize the geometric nature of the construction and we will avoid all technical details. Actually the details of the construction and proofs are quite involved and we refer the reader to [10] for a detailed exposition.

In the first step we construct a continuous function $\gamma\in W^{1,n}$ on $\mathbb{R}^n$ with compact support contained in $B(0,1)$, $\gamma(0)=0$. The function is actually C^∞ smooth except at the origin, where it has accumulating oscillations with height gradually vanishing at 0.

Next we replace a subset of S^n diffeomorphic to $B(0,1)$ with the graph of γ. The resulting space denoted by S_∞ is homeomorphic to S^n, and actually diffeomorphic everywhere but at one point. S_∞ is constructed as a subset of $\mathbb{R}^{n+1}$. Since γ belongs to $W^{1,n}$ there is a $W^{1,n}$ homeomorphism f of S^n onto S_∞.

It turns out that Lipschitz mappings $\mathrm{Lip}\,(S^n,S_\infty)$ are not dense in $W^{1,n}(S^n,S_\infty)$. Indeed, homotopy properties of $W^{1,n}$ mappings imply that if a Lipschitz mapping $g\in\mathrm{Lip}\,(S^n,S_\infty)$ is sufficiently close to f, in the Sobolev norm, then g must be a surjective mapping. On the other hand the oscillations of γ are so frequent that there is no Lipschitz surjection $g:S^n\to S_\infty$.

Thus there is $\varepsilon>0$ such that

$$\|f-g\|_{1,n}>\varepsilon \text{ for all } g\in\mathrm{Lip}\,(S^n,S_\infty). \tag{4.1}$$

The function γ is defined as a series

$$\gamma=\sum_{i=1}^\infty \eta_i,$$

where η_k are a smooth, compactly supported bump functions. Let $\widetilde{S}_k$ be the manifold obtained from S^n be replacing a subset diffeomorhphic to $B(0,1)$

with the graph of

$$\gamma_k = \sum_{i=1}^{k} \eta_i.$$

Clearly $\widetilde{S}_k \subset \mathbb{R}^{n+1}$ is a smooth manifold diffeomorphic to S^n and the sequence $\widetilde{S}_k$ converges in some sense to S_∞. Each set $\widetilde{S}_k$ and S_∞ is a subset of $\mathbb{R}^{n+1} \subset \mathbb{R}^{n+2} = \mathbb{R}^{n+1} \times \mathbb{R}$. Let S_k be the translation of $\widetilde{S}_k$ by the vector $\langle 0, \dots, 0, 2^{-k} \rangle$ in $\mathbb{R}^{n+2}$ and now we define

$$\widetilde{X} = S_\infty \cup \bigcup_{k=1}^{\infty} S_k.$$

Thus the set $\widetilde{X}$ consists of countably many slices. Slices S_k are smooth manifolds diffeomorphic to S^n and they converge to the limiting slice S_∞. The set $\widetilde{X}$ is compact, but not connected. To make it connected we define X by adding to $\widetilde{X}$ a curve that connects all the sets in the family and has the property that no part of the curve is rectifiable.

Using absolute continuity of Sobolev mappings on lines one can easily prove the following fact.

Lemma 4.1. *For each $f \in W^{1,n}(S^n, X)$ we can choose a representative (in the class of functions equal a.e.) such that $f(S^n) \subset S_k$ for some $k = 1, 2, \dots$ or $k = \infty$.*

This implies the following result; for a detailed proof, see [10].

Lemma 4.2. *Lipschitz mappings $\mathrm{Lip}\,(S^n, X)$ are dense in $W^{1,n}(S^n, X)$. Moreover there is $f \in W^{1,n}(S^n, X)$ and $\varepsilon > 0$ such that if $g \in \mathrm{Lip}\,(S^n, X)$, $\|f - g\|_{1,n} < \varepsilon$, then $f(x) \neq g(x)$ for all $x \in S^n$.*

The idea of the proof is as follows. If $f(S^n) \subset S_k$ for a finite k, then f can be approximated by smooth mappings $f_i \in C^\infty(S^n, S_k)$. This is known and follows from the fact that S_k is a smooth manifold. If $f(S^n) \subset S_\infty$, then we can "push" the mapping a little bit, to obtain a mapping $f_k \in W^{1,n}(S^n, S_k)$. Since the manifolds S_k converge to S_∞, the mappings f_k converge to f in the Sobolev norm. Now each mapping f_k can be approximated by mappings in $C^\infty(S^n, S_k)$ (as explained earlier) and hence we obtain not only Lipschitz, but even a smooth approximation of f.

Let $f : S^n \to S_\infty \subset X$ be a $W^{1,n}$ homeomorphism and let $\varepsilon > 0$ be as in (4.1). If $g \in \mathrm{Lip}\,(S^n, X)$ is such that $\|f - g\|_{1,n} < \varepsilon$, then g cannot be a mapping into S_∞, so it must be a mapping into S_k for some finite k and hence $f(x) \neq g(x)$ for all $x \in S^n$.

Now we are ready to complete the proof of Theorem 1.3.

Let $\kappa : X \to \ell^\infty$ be the Kuratowski embedding. Let $f : S^n \to S_\infty \subset X$ be a $W^{1,n}$ homeomorphism. We can, of course, assume that $\nabla f \neq 0$ a.e. Then $\bar{f} = \kappa \circ f \in W^{1,n}_\kappa(S^n, X)$. Suppose that $\bar{g}_k \in \mathrm{Lip}\,(S^n, \kappa(X))$ converge to $\bar{f}$ in the Sobolev norm, $\|\bar{f} - \bar{g}_k\|_{1,n} \to 0$. Define $g_k \in \mathrm{Lip}\,(S^n, X)$ by $g_k = \kappa^{-1} \circ \bar{g}_k$. It follows from Corollary 2.7 that

$$|\nabla(\bar{f} - \bar{g}_k)| \geq C\,(|\nabla f| + |\nabla g_k|)\,\chi_{\{f \neq g_k\}}\,.$$

(Actually one needs a slight modification of the corollary, because now we consider the Kuratowski embedding of X and not of the entire ambient space $\mathbb{R}^{n+2}$ of which X is a subset, but the argument remains the same – we modify the proof of Lemma 2.6 by choosing points x_i and $x_{i'}$ from X; we leave details to the reader.) Hence

$$0 \leftarrow \|\bar{f} - \bar{g}_k\|^n_{1,n} \geq C \left(\int_{\{f \neq g_k\}} |\nabla f|^n + \int_{\{f \neq g_k\}} |\nabla g_k|^n \right).$$

Thus $|\{f \neq g_k\}| \to 0$ and

$$\int_{\{f \neq g_k\}} |\nabla g_k|^n \to 0\,.$$

This, in turn, implies that $g_k \to f$ in $W^{1,n}(S^n, X)$. Therefore Lemma 4.2 implies that for all sufficiently large k, $g_k \neq f$ everywhere, which contradicts the fact that $|\{f \neq g_k\}| \to 0$. The proof is complete. □

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